Def (tower)

G = group

- A tower of G is just a sequence of sub gps:

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n$$

- A tower of G is called normal, if

$$G_i \triangleright G_{i+1}, \qquad \forall i$$

- A tower of G is called abelian, if

  it is normal and G_i / G_{i+1} is abelian for all i.

- A tower of G is called cyclic, if

  it is abelian and furthermore each G_i / G_{i+1} is cyclic.

Def (solvable group)

A group G is called solvable, if there is an abelian tower of G such that the last element $G_n = \{0\}$.

Example (1) $\{$cyclic groups$\} \subsetneq \{$abelian groups$\} \subsetneq \{$solvable groups$\}$

$\mathbb{Z}\times\mathbb{Z}$ is abelian, but not cyclic

S_3 is solvable, but not abelian.

$S_3 = \langle \sigma, \tau \rangle$, where $\sigma^3 = e$, $\tau^2 = e$.

Then $\langle \sigma \rangle \lhd S_3$, and $S_3 / \langle \sigma \rangle \simeq \mathbb{Z}_2$.

Thus: $S_3 \rhd \langle \sigma \rangle \rhd \{e\}$ is an abelian tower.

(2). $B_n = \{$ upper triangular $n\times n$ matrices with non zero det$\}$

$U = \{ (a_{ij}) \in GL_n(k) \mid a_{ij} = 0, \; i > j \}$

$$
\begin{aligned}
B_{n,0} &= \{ (a_{ij}) \in B_n \mid a_{ii} = 1, \forall i \} \\
&\cup \\
B_{n,1} &= \{ (a_{ij}) \in B_{n,0} \mid a_{i,i+1} = 0, \forall i \} \\
&\vdots \\
&\cup \\
B_{n,k} &= \{ (a_{ij}) \in B_{n,k-1} \mid a_{i,\, i+k} = 0, \forall i \} \\
&\vdots \\
&\cup \\
B_{n,n-1} &= \{ (a_{ij}) \in B_{n,n-2} \mid a_{i,\, i+n-1} = 0, \forall i \} \\
&= \{e\}
\end{aligned}
$$

Check: The above tower is abelian.

Prop: $G' = G'_0 \supset \cdots \supset G'_n$ is a(n) cyclic (~~resp~~ resp. abelian) tower. Then for $\forall\, f : G \to G'$ homo.

$$G = \underset{\substack{\| \\ G_0}}{f^{-1}(G')} = \cancel{G_0} \supset \underset{\substack{\| \\ G_1}}{f^{-1}(G'_1)} \supset \cdots \supset \underset{\substack{\| \\ G_n}}{f^{-1}(G'_n)}$$

is a cyclic (resp. abelian) tower of G.

Pf: Note f induces monomorphism:

$$\frac{f^{-1}(G'_i)}{f^{-1}(G'_{i+1})} \overset{\bar{f}}{\hookrightarrow} \frac{G'_i}{G'_{i+1}}$$

Then it is easy to see that a subgp of a cyclic (resp. abelian) gp is again cyclic (resp abelian).

#

Def (Simple group)

A group is simple, if it is nontrivial and has ~~only~~ trivial normal subgroup.

That is, $N \triangleleft G \Rightarrow N = \{e\}$ or $N = G$.

Example: G: abelian group. Then G is simple iff (= if and only if)

$|G| = p$, prime number.

pf: $(\Leftarrow)$ Assume $|G| = p$, (then $G = \mathbb{Z}/p\mathbb{Z}$)

$N \triangleleft G \Rightarrow |N| \,\big|\, |G| \Rightarrow |N| = 1$ or $|N| = p$

$\Updownarrow$

($N \leq G$, since G abelian) $\Rightarrow N = \{e\}$ or $N = G$.

Thus G is simple

$(\Rightarrow)$ prove by contradiction. Assume $|G| \neq p$, for any prime p.

~~Then $\exists x \in G$, $x \neq e$, $ord(x) = \infty$ or $ord(x) = n$~~

Take any $x \neq e$ in G. Then G simple $\Rightarrow$

$G = \langle x \rangle$

Case 1: $G \cong \mathbb{Z}$.

Then $\langle x^2 \rangle \neq \langle x \rangle$ is a nontrivial normal subgp. ↯

Case 2: $G \cong \mathbb{Z}/n\mathbb{Z}$, $n = \prod_{i=1}^{s} p_i^{r_i}$. $r_i \geq 1$,

Take any $p \mid n$. Then $n = p \cdot n'$. $n' \neq 1$.

Then $\langle x^{n'} \rangle \neq \langle x \rangle$ is a nontrivial normal subgp. ↯

#

Famous example of simple nonabelian groups

(1) A_n, $n \geq 5$.

(alternating group)

Let $\{e_1, \cdots, e_n\}$ be the standard basis of k^n.

Define

$$\begin{array}{ccc} S_n & \xrightarrow{\phi} & \{\pm 1\} \overset{\shortparallel}{} \mu_2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma & \longmapsto & \det\left(e_{\sigma(1)}, \dots, e_{\sigma(n)}\right) \end{array}$$

(Ex): ϕ is an epimorphism.

$A_n \stackrel{\circ}{=} \ker(\phi)$

Thm: A_n is simple, if $n \geq 5$.

Rmk: Another way to define ϕ is $\dfrac{\prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}$.

Q: why these two definitions are the same?

(2) $PSL_n(K)$, $n \geq 2$.

$$PSL_n(K) = SL_n(K) / Z(SL_n(K)) = SL_n(K) / \mu_n(K)$$

Rmk: Simple groups are the atom objects in the category of groups.
Therefore, a classification of them is a very important question.

(Simple Lemma): $f: G \longrightarrow G'$ homo with G simple.
Then ~~if~~ f is either injective or $f(G) = e'$ (i.e. f is trivial).

The importance of simple groups also shows up in the following

Theorem (Jordan-Hölder)

(1) Any finite group G admits a normal tower

$$G = G_0 \supset G_1 \supset \cdots \supset G_r \supset G_{r+1} = \{e\}$$

Such that $\left\{ G_i/G_{i+1},\ 0 \leq i \leq r \right\}$ are simple groups

(2) Let $G = G_0 \supset G_1 \supset \cdots \supset G_r \supset G_{r+1} = \{e\}$ be a normal tower with $\left\{ G_i/G_{i+1},\ 0 \leq i \leq r \right\}$ simple.

Assume $G = H_0 \supset H_1 \supset \cdots \supset H_s \supset H_{s+1} = \{e\}$, be another normal tower with $\left\{ H_j/H_{j+1},\ 0 \leq j \leq s \right\}$ simple.

Then $r = s$, and, up to isomorphisms,

$$\left\{ G_i/G_{i+1} \mid 0 \leq i \leq r \right\} = \left\{ H_j/H_{j+1} \mid 0 \leq j \leq s = r \right\}$$

proof of (1).:

step (1) If G is simple, we've done.

step (2). Take $N \lhd G$, nontrivial.

Consider $G \xrightarrow{\pi} G/N$

Do induction on $|G|$. Then by induction hypothesis.

$$G/N = H_0 \supset H_1 \supset \cdots \supset H_r \supset H_{r+1} = \{e\}, \text{ s.t}$$

$$\left\{ H_i / H_{i+1} \right\} \text{ simple}$$

Then we get

$$\begin{array}{ccccccccc} G = G_0 & \supset & G_1 & \supset & \cdots & \supset & G_r & \supset & \overset{N}{\overset{\parallel}{G_{r+1}}} \supset \{e\} \\ \parallel & & \parallel & & & & \parallel & & \parallel \\ \pi^{-1}(H_0) & \supset & \pi^{-1}(H_1) & \supset & & \supset & \pi^{-1}(H_r) & \supset & \pi^{-1}(H_{r+1}) \\ & & & & & & & & \parallel \\ & & & & & & & & \{e\} \end{array}$$

step (2.1). $G_i / G_{i+1} \overset{\overline{\pi}}{\xrightarrow{\sim}} H_i / H_{i+1}$.

pf:

$$\begin{tikzcd} G_i = \pi^{-1}(H_i) \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "\overline{\pi}"'] & H_i \arrow[r, two heads] & H_i / H_{i+1} \end{tikzcd}$$

$$\ker(\overline{\pi}) = \{ x \in G_i \mid \pi(x) \in H_{i+1} \}$$

$$= \{ x \in G_i \mid x \in \pi^{-1}(H_{i+1}) \}$$

$$= G_{i+1}. \qquad \#$$

Step (2.2) Note $|N| < |G|$.

Again by induction hypothesis,

$$N = N_0 \supset N_1 \supset \cdots \supset N_{r'} \supset N_{r'+1} = \{e\}.$$

with $\left\{ N_j / N_{j+1} \right\}$ simple

Thus

$$\begin{array}{ccccccccccccccc} & & & & & & & N & \supset & N_1 & \supset \cdots \supset & N_{r'} & \supset & N_{r'+1} & \\ & & & & & & & \| & & \| & & \| & & \| & \\ G = G_0 & \supset & G_1 & \supset \cdots & \supset & G_r & \supset & G_{r+1} & \supset & G_{r+2} & \supset \cdots \supset & G_{r+r'+1} & \supset & \{e\} & \\ & & & & & & & & & & & & & \| & \\ & & & & & & & & & & & & & G_{r+r'+2} & \end{array}$$

is a normal tower, with

$$\left\{ G_i / G_{i+1}, \quad 0 \le i \le r+r'+1 \right\} \text{ simple}$$

#

Proof of (2):

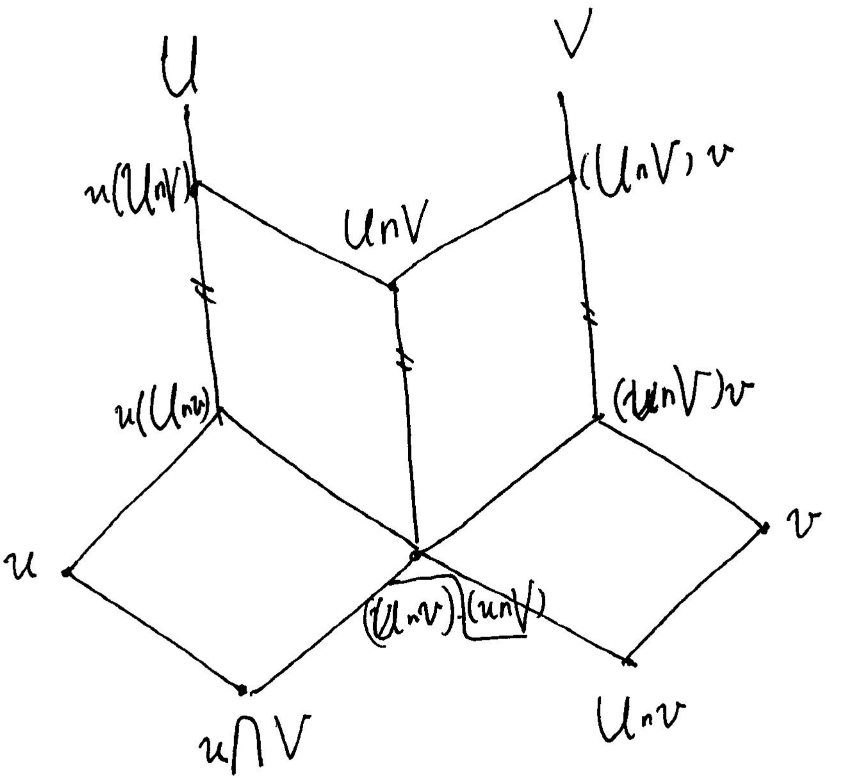
Step (1) (Butterfly Lemma)

$U, V \le G$, $u \triangleleft U$, $v \triangleleft V$. Then

(1) $u(U \cap v) \triangleleft u(U \cap V)$

$(u \cap V)v \triangleleft (U \cap V)v$

(2) $$\frac{u(U \cap V)}{u(U \cap v)} \simeq \frac{(U \cap V)v}{(u \cap V)v}$$



Pf: Show $$\frac{u(U\cap V)}{u(U\cap v)} \cong \frac{U\cap V}{(u\cap V)\cdot(U\cap v)} \cong \frac{(U\cap V)v}{(u\cap V)v}$$

Use $H = U\cap V$, $K = u(U\cap v)$

Then $H \leq N_K$

$HK = KH = u(U\cap V)$

$H\cap K = (u\cap V)(U\cap v)$ (Exercise!)

Then $HK/K \cong H/H\cap K$ gives the first iso.

The second iso is obtained by symmetry.

#

Step (2) (Schreier)

A refinement of a ~~normal~~ tower is to insert a finite number of subgroups in the given tower.

Let $G = G_0 \supset G_1 \supset \cdots \supset G_r \supset G_{r+1} = \{e\}$ be a normal tower

and $G = H_0 \supset H_1 \supset \cdots \supset H_s \supset H_{s+1} = \{e\}$ be another normal tower.

Then these two normal towers have an "equivalent" refinement.

pf: For $1 \le i \le r$, $1 \le j \le s$, define

$$G_{ij} = G_{i+1}(H_j \cap G_i)$$

$$H_{ji} = H_{j+1}(G_i \cap H_j)$$

Then

$$\begin{array}{l}
G = \overset{G_1}{\overset{\parallel}{G_{11}}} \supset G_{12} \supset \cdots \supset G_{1,s} \supset \overset{G_2}{\overset{\parallel}{G_{1,s+1}}} \supset \\
G_{21} \supset G_{22} \supset \cdots \supset G_{2,s} \supset \overset{G_3}{\overset{\parallel}{G_{2,s+1}}} \supset \\
\vdots \\
G_{r,1} \supset \quad \cdots \supset G_{r,s} \supset \overset{G_r}{\overset{\parallel}{G_{r,s+1}}} \\
H_1 \cap G_r = G_{r+1,1} \supset \cdots \supset \overset{H_s \cap G_r}{\overset{\parallel}{G_{r+1,s}}} \supset \overset{\{e\}}{\overset{\parallel}{G_{r+1,s+1}}}
\end{array}$$

Similarly for H. Note that Step (1) $\Rightarrow$

$$G_{ij}/G_{i,j+1} \cong H_{j,i}/H_{j,i+1}$$

#

Step (3) Lemma:

$N \triangleleft G$, s.t G/N single.

~~Let~~ $G \triangleleft H \triangleleft N$ be a refinent of the tower $G \triangleright N$.

Then $H = G$ or $H = N$.

pf: Exercise.

Now, we assume in step (2), the normal towers have the properties that

$$\left\{ G_i/G_{i+1} \right\}_{1 \le i \le r}, \qquad \left\{ H_j/H_{j+1} \right\}_{1 \le j \le s} \quad \text{are single.}$$

Then Step(2) shows.

$$\begin{array}{ccc} \{G_i\} & < & \{G_{ij}\} \\ & \text{refinent} & \wr\wr \\ \{H_j\} & < & \{H_{ji}\} \end{array}$$

But by the lemma, $\forall i,\ \exists!\ j = j(i)$, s.t

$$G_i/G_{i+1} = G_{ij}/G_{i,j+1} \cong H_{ji}/H_{j,i+1} = H_j/H_{j+1}$$

The theorem is proved.

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